Journal of Engineering Mathematics **32:** 305–320, 1997.

Harmonic waves in a prestressed thin elastic tube filled with a viscous fluid

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Received 3 August 1995; accepted in revised form 23 July 1996

Abstract. In this work a theoretical analysis is presented for wave propagation in a thin-walled prestressed elastic tube filled with a viscous fluid. The fluid is assumed to be incompressible and Newtonian, whereas the tube material is considered to be incompressible, isotropic and elastic. Considering the physiological conditions that the arteries experience, such a tube is initially subjected to a mean pressure P_i and an axial stretch λ_z. If it is assumed that in the course of blood flow small incremental disturbances are superimposed on this initial field, then the governing equations of this incremental motion are obtained for the fluid and the elastic tube. A harmonic-wave type of solution is sought for these field equations and the dispersion relation is obtained. Some special cases, as well as the general case, are discussed and the present formulation is compared with some previous works on the same subject.

Key words: tube, elasticity, waves, blood flow, prestressed.

1. Introduction

Propagation of harmonic waves in an initially stressed (or unstressed) circular cylindrical tube filled with a viscous (or inviscid) fluid has been a problem of interest since the time of Thomas Young who first obtained the speed of pulse waves in human arteries. The current literature on the subject is so rich that it is almost impossible to cite all the contributions here. The historical evolution of the subject may be found in the books by McDonald [1] and Fung [2] and in the papers by Lambossy [3] and Skalak [4]. Significant contributions to the understanding of wave motions in elastic tubes filled with a viscous fluid have been made by Witzig [5], Morgan and Kiely [6], Womersley [7], Atabek and Lew [8], Mirsky [9], Atabek [10] and more recently by Rachev [11] and Kuiken [12]. In all these works, either the effects of initial stresses have been neglected or taken into account in an *ad hoc* manner. Moreover, the elastic coefficients of the incremental stresses have been treated as constants. In essence, these coefficients are not material constants, but rather are functions of initial deformation.

For a healthy human being, the mean blood pressure is around 100 mm Hg and the axial stretch of artery is about 1·5. This means that the inner pressure and the axial stretch create relatively large circumferential and axial stresses. On the other hand, in the course of blood flow, the pressure deviation exerted by the left ventricle is around ± 20 mm Hg. Considering the stiffening properties of soft biological tissues with stress, the dynamical deformation resulting from this pressure deviation may be assumed to be small compared to the initial static deformation. Therefore, the theory of small deformations superimposed on large initial static deformation may be utilized in analyzing wave propagation in such a composite structure.

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In this work, utilizing the mechanical model proposed by Demiray [13], in which the arterial wall material is assumed to be incompressible, isotropic and elastic, we study the propagation of harmonic waves in an elastic thin tube filled with an incompressible viscous fluid. Considering the physiological conditions that the arteries experience first, we obtain the stress distribution (or stress resultant) under the effect of a uniform inner pressure and a constant axial stretch. Superimposition of a small, but dynamic displacement field over this static deformation allows the governing incremental equations of motion for both fluid and tube to be obtained when cylindrical polar coordinates are used. Seeking a harmonic-wave type of solution to the field equations of the fluid and tube, and then using appropriate boundary conditions, we obtain the dispersion relation. Various special cases are discussed and the results are compared with previous studies on the same subject. Owing to the difficult nature of the analysis of the general dispersion relation, a numerical technique has been employed and the variations of propagation speed and transmission coefficients with Womersley parameter and stretch ratios are evaluated. The results are depicted in graphical form.

2. Basic equations

Due to the interaction of blood with its container, the pulsatile motion of the heart leads to wave phenomena both in blood and arteries. The governing field equations and the boundary conditions should, therefore, include these interactions.

2.1. Equations for a fluid

Depending on the scale of strain rates, blood behaves like a Newtonian and/or non-Newtonian incompressible fluid. In the course of pulsatile flow in arteries, blood is subjected to a large uniform pressure P_i and small velocity and pressure increments are added to this initial field. This incremental behavior of blood may then be treated as an incompressible Newtonian fluid. For axially symmetric motion, the governing differential equations in cylindrical polar coordinates are given by

$$\begin{aligned} &-\frac{\partial \bar{p}}{\partial r} + \mu\left(\frac{\partial^2 \bar{u}}{\partial r^2} + \frac{1}{r}\frac{\partial \bar{u}}{\partial r} - \frac{\bar{u}}{r^2} + \frac{\partial^2 \bar{u}}{\partial z^2}\right) - \bar{\rho}\frac{\partial \bar{u}}{\partial t} = 0, \\ &-\frac{\partial \bar{p}}{\partial z} + \mu\left(\frac{\partial^2 \bar{w}}{\partial r^2} + \frac{1}{r}\frac{\partial \bar{w}}{\partial r} + \frac{\partial^2 \bar{w}}{\partial z^2}\right) - \bar{\rho}\frac{\partial \bar{w}}{\partial t} = 0 \end{aligned} \tag{2.1}$$

and the incompressibility equation

$$\frac{\partial \bar{u}}{\partial z} + \frac{\bar{u}}{r} + \frac{\partial \bar{w}}{\partial z} = 0, \tag{2.2}$$

where $\bar{\rho}$ is the mass density, μ is the viscosity, $\bar{p}$ is the incremental pressure, $\bar{u}$ and $\bar{w}$ are incremental velocity components in the radial and axial directions, respectively. The stress components which we need when applying the boundary conditions are given by

$$\bar{\sigma}_{rr} = -\bar{p} + 2\mu\frac{\partial \bar{u}}{\partial r}, \qquad \bar{\sigma}_{rz} = \mu\left(\frac{\partial \bar{u}}{\partial z} + \frac{\partial \bar{w}}{\partial r}\right). \tag{2.3}$$

2.2. Equation for a solid body

The arterial wall material is known to be incompressible, anisotropic and viscoelastic (see Fung *et al*. [2] and Cox [14]). For its simplicity in nonlinear analysis, the arterial wall material shall be assumed to be incompressible, homogeneous, isotropic and elastic. A set of stress-strain relations for such a material was proposed previously by Demiray [13] as

$$t_{kl} = P\delta_{kl} + \beta\exp[\alpha(I-3)]B_{kl}, \qquad B_{kl} = I_1 c_{kl} - c_{km}c_{ml}, \tag{2.4}$$

where t_{kl} is the stress tensor, P is the hydrostatic pressure to be determined from the field equations and the boundary conditions, α and β are two material coefficients to be determined from experimental measurements, $I = \frac{1}{2}B_{kk}$ and $I_1 = c_{kk}$ is the first invariant of the Finger deformation tensor c_{kl}, defined by

$$c_{kl} = F_{kK}F_{lK}. \tag{2.5}$$

Here $F_{kK} = \partial x_k/\partial X_K$ is the gradient of deformation $x_k = x_k(X_K)$ and the summation convention applies to repeated indices. The stress tensor must satisfy Cauchy's equations of equilibrium

$$t_{kl;k} = 0. \tag{2.6}$$

Here the indices following a semi-colon denote the covariant (or contravariant) differentiation with respect to those indices.

Now let us consider a circular cylindrical thin tube made of such an isotropic, incompressible and elastic material subjected to a large inner pressure P_i and an axial stretch λ_z. Upon application of such a symmetrical loading, an axially symmetric deformation field will be developed in the body. Considering the incompressibility of the material, we may describe the deformation in cylindrical coordinates by

$$r = (R^2/\lambda_z + C)^{1/2}, \quad \theta - \Theta, \quad z = Z\lambda_z, \tag{2.7}$$

where C is an integration constant to be determined from the boundary conditions. The physical components of the tensor B_{ij} may be given by

$$[\mathbf{B}] = \begin{bmatrix} \lambda_\theta^{-2} + \lambda_z^{-2} & 0 & 0 \\ 0 & \lambda_z^{-2} + \lambda_\theta^2\lambda_z^2 & 0 \\ 0 & 0 & \lambda_\theta^{-2} + \lambda_\theta^2\lambda_z^2 \end{bmatrix},$$

$$I = \lambda_\theta^{-2} + \lambda_z^{-2} + \lambda_\theta^2\lambda_z^2, \quad \lambda_\theta \equiv r/R. \tag{2.8}$$

Introducing (2.8) into (2.4) and then substituting the result in (2.6) we may give the stress components satisfying the equilibrium equations by

$$\begin{aligned} &t_{rr}^0 = \beta\int_{\lambda_\theta}^{\lambda_\theta^o} F(x)(1+\lambda_z x^2)x^{-3}\,\mathrm{d}x, \qquad t_{\theta\theta}^0 = t_{rr}^0 + \beta F(\lambda_\theta)(\lambda_\theta^2\lambda_z^2 - \lambda_\theta^{-2}), \\ &t_{zz}^0 = t_{rr}^0 + \beta F(\lambda_\theta)(\lambda_\theta^2\lambda_z^2 - \lambda_z^{-2}), \qquad t_{kl} = 0(k \neq 1), \end{aligned} \tag{2.9}$$

Table 1. Pressure-radii relation for an artery

$P_i(Pa)$ exper.	$r_i \times 10^2$ (m)	$r_0 \times 10^2$ (m)	$P_i(Pa)$ theor.	Deviation
3,350	0·348	0·401	3,275	$-1{\cdot}7$
6,670	0·396	0·445	6,580	$-1{\cdot}4$
10,000	0·425	0·473	9,938	$-0{\cdot}6$
13,340	0·442	0·485	13,830	$+3{\cdot}6$
20,000	0·467	0·510	19,886	$-0{\cdot}6$
26,680	0·485	0·524	25,970	$-2{\cdot}7$

At zero stress $r_i = 0{\cdot}31 \times 10^{-2}$ m, $r_0 = 0{\cdot}38 \times 10^{-2}$ m

with

$$F(\lambda_\theta) \equiv \exp[\alpha(\lambda_\theta^{-2} + \lambda_z^{-2} + \lambda_\theta^2\lambda_z^2 - 3)]. \qquad (2.10)$$

From (2.9) the inner pressure is expressed as

$$P_i = \beta \int_{\lambda_\theta^o}^{\lambda_\theta^i} F(x)(1 + \lambda_z x^2)x^{-3})\, \mathrm{d}x, \qquad (2.11)$$

where λ_θ^i and λ_θ^o stand for the values of the circumferential stretch ratios on the inner and outer surfaces of the cylinder, respectively.

Equation (2.11) relates the deformation of the tube to the inner pressure P_i. If the inner pressure-radius relation is known experimentally, by comparing theoretical results with experimental measurements we can determine the material constants α and β numerically so as to obtain the best fit between the experiment and the theoretical model

Simon *et al*. [15] conducted experiments on canine abdominal arteries and measured the inner pressure-radii relations, listed in Table I for $\lambda_z = 1{\cdot}53$. Using the least-squares method, for the best fit of the theoretical model to the experimental measurements, we determine the values of material coefficients as $\alpha = 0{\cdot}82$ and $\beta = 10{\cdot}1 \times 10^3$ Pa. Employing these numerical coefficients in (2.11), we may calculate the theoretical pressures. These are listed in the fourth column of the table. The deviation between experiment and the model is given in the fifth column of the same table. The result reveals that the maximum deviation between theory and experiment is about 3·6 percent, which seems to be a fairly good approximation. Therefore, the constitutive relation given in (2.4) may also be used as a fairly good model to describe the mechanical behavior of arterial wall material.

Having determined the initial static deformation and the associated initial stress t_{kl}^0 upon this field, we shall now superimpose a small axially symmetric displacement field $\mathbf{u}(z,t)$, or, in component form, $u_1 = u(z,t), u_2 = 0, u_3 = w(z,t)$, where u and w are the incremental displacement components in the radial and axial directions, respectively. Let r_0 be the radial coordinate of the midsurface of the tube after this finite initial deformation. A material point located at (r_0, θ, z) on the midsurface will move to a new position $(r_0 + u, \theta, z + w)$ after superimposition of such a small displacement field. Thus, the position vector of this point after this final deformation will be

$$\mathbf{r} = [r_0 + u(z,t)]\mathbf{e}_r + [z + w(z,t)]\mathbf{e}_z, \qquad (2.12)$$

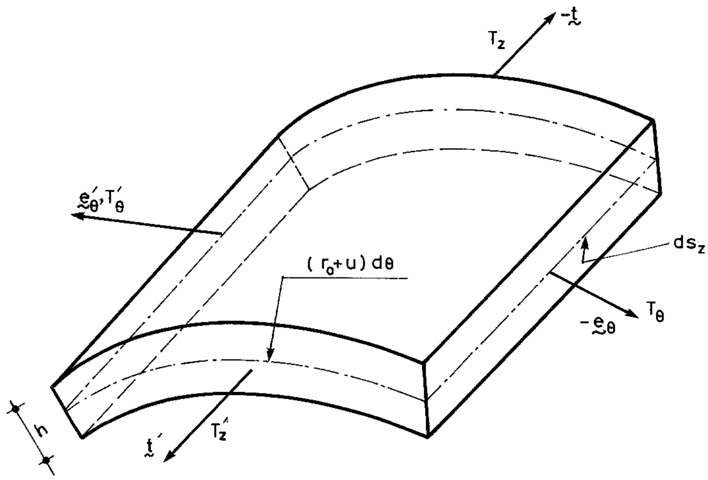

Figure 1. Forces acting on a small membrane element.

where $(\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z)$ are the base vectors in cylindrical polar coordinates. The vector along the generator of the cylinder transforms into

$$\mathbf{T}_z = \frac{\partial \mathbf{r}}{\partial z} = u_{,z}\mathbf{e}_r + (1 + w_{,z})\mathbf{e}_z; \quad \frac{\partial(\cdot)}{\partial z} \equiv_{,z} . \tag{2.13}$$

The unit tangent vector is then defined by

$$\mathbf{t}_z = \mathbf{T}_z/|\mathbf{T}_z| = u_{,z}\mathbf{e}_r + \mathbf{e}_z. \tag{2.14}$$

In obtaining this expression we assumed the incremental displacements and their gradients to be small.

The external unit normal vector $\mathbf{n}$ to the deformed midsurface of the tube is given by

$$\mathbf{n} = \mathbf{e}_\theta \times \mathbf{t}_z = \mathbf{e}_r - u_{,z}\mathbf{e}_z. \tag{2.15}$$

Now, let us consider a tube element placed between the planes $\theta = \text{const.}, \theta + \mathrm{d}\theta = \text{const.}, z =$ const. and $z + \mathrm{d}z = \text{const.}$ (Figure 1). The elementary arc lengths $\mathrm{d}s_z$ and $\mathrm{d}s_\theta$ are defined by

$$\mathrm{d}s_z = |\mathbf{T}_z| dz \cong (1 + w_{,z})\,\mathrm{d}z$$

$$\mathrm{d}s_\theta = (r_0 + u)\,\mathrm{d}\theta. \tag{2.16}$$

Similarly, the elementary area of this element on the midsurface is given as

$$\mathrm{d}a \cong (r_0 + r_0 w_{,z} + u)\,\mathrm{d}z\,\mathrm{d}\theta. \tag{2.17}$$

In order to write the equations of motion of the tube we need to know the total force acting on this element. Let the external force acting per unit area of the inner surface be represented by

$$\mathbf{P} = P_t\mathbf{t}_z + P_n\mathbf{n}. \tag{2.18}$$

If the membrane forces along the vectors $\mathbf{t}_z$ and $\mathbf{e}_\theta$ are denoted by N_z and N_θ, then the total force acting on this element is given by

$$\mathbf{F} = \frac{\partial}{\partial z}[N_z\mathbf{t}_z(r_0+u)]\mathrm{d}\theta\,\mathrm{d}z - N_\theta\mathbf{e}_r(1+w_{,z})\,\mathrm{d}\theta\,\mathrm{d}z + (P_t\mathbf{t}_z + P_n\mathbf{n})(r_0 + r_0 w_{,z} + u)\,\mathrm{d}\theta\,\mathrm{d}z. \tag{2.19}$$

Equating this force to mass times the acceleration of the element, which is given by

$$\rho h(r_0 + r_0 w_{,z} + u)\,\mathrm{d}\theta\,\mathrm{d}z(u_{,tt}\mathbf{e}_r + w_{,tt}\mathbf{e}_z) \tag{2.20}$$

and dividing both sides of the corresponding equation by $(r_0 + r_0 w_{,z} + u)\,\mathrm{d}\theta\,\mathrm{d}z$, we obtain

$$(r_0 + r_0 w_{,z} + u)^{-1}[N_z u_{,z}\mathbf{t}_z + (r_0+u)(N_z\mathbf{t}_z)_{,z} - (1+w_{,z})N_\theta\mathbf{e}_r] + (P_t\mathbf{t}_z + P_n\mathbf{n}) = \rho h(u_{,tt}\mathbf{e}_r + w_{,tt}\mathbf{e}_z). \tag{2.21}$$

These equations are still nonlinear, but they can be linearized by the introduction of

$$N_z = N_z^0 + \Sigma_z, \quad N_\theta = N_\theta^0 + \Sigma_\theta, \quad |\Sigma_z/N_z^0| \ll 1, |\Sigma_\theta/N_\theta^0| \ll 1,$$
$$P_t = 0 + \bar{P}_t, \quad P_n = P_i + \bar{P}_n, \quad |\bar{P}_n/P_i| \ll 1, \tag{2.22}$$

where P_i is the initial inner pressure, N_θ^0, N_z^0 are initial stress resultants, $\Sigma_\theta, \Sigma_z, P_t$ and $\bar{P}_n$ are the small increments added on these initial fields. Introducing (2.22) into (2.21) and keeping in mind the smallness of the superimposed displacement components and their gradients, we may obtain the linearized field equations as follows

$$N_z^0\frac{\partial^2 u}{\partial z^2} + \frac{N_\theta^0}{r_0^2}u - \frac{\Sigma_\theta}{r_0} + \bar{P}_n = \rho h\frac{\partial^2 u}{\partial t^2}, \tag{2.23}$$

$$\left(\frac{N_z^0 - N_\theta^0}{r_0}\right)\frac{\partial u}{\partial z} + \frac{\partial \Sigma_z}{\partial z} + \bar{P}_t = \rho h\frac{\partial^2 w}{\partial t^2}. \tag{2.24}$$

In obtaining these equations we have used the following equilibrium equation, which is known as Laplace's law,

$$N_\theta^0 = r_0 P_i. \tag{2.25}$$

Here $\bar{P}_n, \bar{P}_t$ are to be determined from the reaction of a viscous fluid with the tube wall and can be expressed as

$$\bar{P}_n = \left(\bar{p} - 2\mu\frac{\partial\bar{u}}{\partial r}\right)_{r=r_0}, \quad \bar{P}_t = -\mu\left(\frac{\partial\bar{u}}{\partial z} + \frac{\partial\bar{w}}{\partial r}\right)_{r=r_0}. \tag{2.26}$$

In order to complete the field equations we must know the constitutive relations for Σ_θ and Σ_z. We recall the definition of membrane forces in the final configuration

$$T'_\theta = h' t'_{\theta\theta}, \quad T'_z = h' t'_{zz}, \tag{2.27}$$

where h' is the thickness, $t'_{\theta\theta}$ and t'_{zz} are the total circumferential and axial stress components in the final configuration. The incompressibility of the material requires that

$$h'(r_0 + u)(1 + w_{,z}) = r_0 h$$

or

$$h' = h(1 + u/r_0)^{-1}(1 + w_{,z})^{-1} \cong h(1 - u/r_0 - w_{,z}). \tag{2.28}$$

Setting $t'_{\theta\theta} = t^0_{\theta\theta} + \bar{t}_{\theta\theta}, t'_{zz} = t^0_{zz} + \bar{t}_{zz}$ in (2.27) and utilizing the approximation (2.28), we have

$$T'_\theta = ht^0_{\theta\theta} + h\left[\bar{t}_{\theta\theta} - t^0_{\theta\theta}(u/r_0 + w_{,z})\right] = N^0_\theta + \Sigma_\theta,$$

$$T'_z = ht^0_{zz} + h\left[\bar{t}_{zz} - t^0_{zz}(u/r_0 + w, z)\right] = N^0_z + \Sigma_z. \tag{2.29}$$

Considering the definitions $N^0_\theta = ht^0_{\theta\theta}$ and $N^0_z = ht^0_{zz}$, from (2.30), we can write

$$\Sigma_\theta = h\left[\bar{t}_{\theta\theta} - t^0_{\theta\theta}(u/r_0 + w_{,z})\right], \qquad \Sigma_z = h\left[\bar{t}_{zz} - t^0_{zz}(u/r_0 + w_{,z})\right]. \tag{2.30}$$

To determine the explicit expressions of Σ_θ and Σ_z we have to know the incremental constitutive equations for $\bar{t}_{kl}$, for which we consult to the theory of so-called 'small-displacements superimposed on large initial static deformation'. The derivation of the field equations and incremental constitutive relations can be found in the books by Green and Zerna [16] and Eringen and Şuhubi [17]. For this particular type of constitutive relations the incremental stress tensor $\bar{t}_{kl}$ referred to the final configuration may be given by

$$\bar{t}_{kl} = \bar{p}\delta_{kl} + \beta \exp[\alpha(I^0 - 3)](\alpha\bar{I}B^0_{kl} + \bar{B}_{kl}), \tag{2.31}$$

where $\bar{p}$ is the increment in hydrostatic pressure, $\bar{I}$ and $\bar{B}_{kl}$ are defined by

$$\bar{I} = 2B^0_{ij}e_{ij}, \quad e_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}),$$

$$\bar{B}_{kl} = u_{k,p}B^0_{pl} + u_{l,p}B^0_{pk} + 2e_{mn}c^0_{mn}c^0_{kl} - 2c_{km}c^0_{nl}e_{mn}. \tag{2.32}$$

Introducing (2.31) into (2.30) we find that the incremental membrane forces are given as follows

$$\Sigma_\theta = h\bar{t}_{\theta\theta} = h\left(\alpha^0_{11}\frac{u}{r_0} + \alpha^0_{12}\frac{\partial w}{\partial z}\right), \qquad \Sigma_z = h\bar{t}_{zz} = h\left(\alpha^0_{21}\frac{u}{r_0} + \alpha^0_{22}\frac{\partial w}{\partial z}\right), \tag{2.33}$$

where the coefficients $\alpha^0_{ij}, (i, j = 1, 2)$ are defined by

$$\alpha^0_{11} = \beta F(\lambda_\theta)\left[(\lambda^2_\theta\lambda^2_z + 3\lambda^{-2}_\theta) + 2\alpha(\lambda^2_\theta\lambda^2_z - \lambda^{-2}_\theta)^2\right],$$

$$\alpha^0_{12} = \beta F(\lambda_\theta)\left[(\lambda^2_\theta\lambda^2_z + \lambda^{-2}_\theta) + 2\alpha(\lambda^2_\theta\lambda^2_z - \lambda^{-2}_z)(\lambda^2_\theta\lambda^2_z - \lambda^{-2}_\theta)\right],$$

$$\alpha^0_{21} = \beta F(\lambda_\theta)\left[(\lambda^2_\theta\lambda^2_z + \lambda^{-2}_z) + 2\alpha(\lambda^2_\theta\lambda^2_z - \lambda^{-2}_z)(\lambda^2_\theta\lambda^2_z - \lambda^{-2}_\theta)\right],$$

$$\alpha^0_{22} = \beta F(\lambda_\theta)\left[(\lambda^2_\theta\lambda^2_z + 3\lambda^{-2}_z) + 2\alpha(\lambda^2_\theta\lambda^2_z - \lambda^{-2}_\theta)^2\right]. \tag{2.34}$$

Introducing (2.34) into (2.23) and (2.24), we have

$$N_z^0 \frac{\partial^2 u}{\partial z^2} + \left(\frac{N_\theta^0 - h\alpha_{11}^0}{r_0^2} \right) u - \frac{\alpha_{12}^0 h}{r_0} \frac{\partial w}{\partial z} + \left(\bar{p} - 2\mu \frac{\partial \bar{u}}{\partial r} \right)_{r=r_0} - \rho h \frac{\partial^2 u}{\partial t^2} = 0,$$

$$\left(\frac{N_z^0 + h\alpha_{21}^0 - N_\theta^0}{r_0} \right) \frac{\partial u}{\partial z} + h\alpha_{22}^0 \frac{\partial^2 w}{\partial z^2} - \mu \left(\frac{\partial \bar{u}}{\partial z} + \frac{\partial \bar{w}}{\partial r} \right)_{r=r_0} - \rho h \frac{\partial^2 w}{\partial t^2} = 0. \tag{2.35}$$

These and Equations (2.1) are to be solved subject to the boundary conditions

$$\bar{u}(r_0, z, t) = \frac{\partial u}{\partial t}, \qquad \bar{w}(r_0, z, t) = \frac{\partial w}{\partial t}. \tag{2.36}$$

3. Solution to field equations

In this section we shall seek a harmonic-wave type of solution to the governing equations given in (2.1) and (2.32). For this purpose we set

$$\{\bar{p}, \bar{u}, \bar{w}; u, w\} = \{\bar{P}(r), \bar{U}(r), \bar{W}(r); A, B\} \exp[i(\omega t - kz)], \tag{3.1}$$

where k is the wave number, ω is the angular frequency, $\bar{P}(r), \bar{U}(r), \bar{W}(r)$ are unknown complex amplitude functions to be determined from the field equations, A and B are two complex constants representing the wave amplitudes of the tube. Introducing (3.1) into (2.1), we obtain

$$-\frac{\mathrm{d}\bar{P}}{\mathrm{d}r} + \mu \left(\frac{\mathrm{d}^2 \bar{U}}{\mathrm{d}r^2} + \frac{1}{r} \frac{\mathrm{d}\bar{U}}{\mathrm{d}r} - \frac{\bar{U}}{r^2} - k^2 \bar{U} \right) - i\bar{\rho}\omega \bar{U} = 0,$$

$$ik\bar{P} + \mu \left(\frac{\mathrm{d}^2 \bar{W}}{\mathrm{d}r^2} + \frac{1}{r} \frac{\mathrm{d}\bar{W}}{\mathrm{d}r} - k^2 \bar{W} \right) - i\bar{\rho}\omega \bar{W} = 0,$$

$$\frac{\mathrm{d}\bar{U}}{\mathrm{d}r} + \frac{\bar{U}}{r} - ik\bar{W} = 0. \tag{3.2}$$

The solution of this set of differential equations may be given by

$$\bar{U} = k[\bar{C} I_1(kr) + \bar{D} J_1(sr)], \bar{P} = -i\bar{\rho}\omega \bar{C} I_0(kr),$$

$$\bar{W} = -i[k\bar{C} I_0(kr) + s\bar{D} J_0(sr)], s^2 \equiv -k^2 - i\bar{\rho}\omega/\mu, \tag{3.3}$$

where $\bar{C}$ and $\bar{D}$ are two integration constants, $I_n(kr)$ and $J_n(sr)$ are modified and first-kind Bessel functions of order n, respectively.

To obtain solutions for the equations of an elastic tube, we introduce (3.1) and (3.3) into (2.35), which results

$$\left(\rho h\omega^2 - N_z^0 k^2 + \frac{N_\theta^0 - h\alpha_{11}^0}{r_0^2} \right) A + \frac{ikh\alpha_{12}^0}{r_0} \bar{B}$$

$$+ \left[-(2\mu k^2 + i\bar{\rho}\omega) I_0(kr_0) + \frac{2\mu k}{r_0} I_1(kr_0) \right] \bar{C} + \left[\frac{2\mu k}{r_0} J_1(sr_0) - 2\mu ks J_0(sr_0) \right] \bar{D} = 0, \tag{3.4}$$

$$-ik\left(\frac{(N_z^0 + h\alpha_{21}^0 - N_\theta^0)}{r_0}\right)A + (\rho h\omega^2 - k^2 h\alpha_{22}^0)B + 2i\mu k^2 I_1(kr_0)\bar{C}$$
$$+(2i\mu k^2 - \bar{\rho}\omega)J_1(sr_0)\bar{D} = 0. \tag{3.5}$$

The boundary conditions (2.36) take the following form

$$i\omega A - KI_1(kr_0)\bar{C} - kJ_1(sr_0)\bar{D} = 0, \tag{3.6}$$

$$\omega B + kI_0(kr_0)\bar{C} + sJ_0(sr_0)\bar{D} = 0. \tag{3.7}$$

Equations (3.4)–(3.6) give four homogeneous algebraic relations between $A, B, \bar{C}, \bar{D}$. In order for us to have a non-trivial solution for these coefficients, the determinant of the coefficient matrix must vanish. Before we write this determinant, we shall first eliminate A and B from these algebraic equations, *i.e*.

$$\left\{\left[m\left(\Omega^2 - S\xi^2 + \frac{G-\alpha_{11}}{\bar{\lambda}_\theta^2}\right) + 2i\nu\Omega\lambda_z\right]\xi^2 f(\xi)\right.$$
$$\left. + \left(\lambda_z\Omega^2 + \frac{\alpha_{12}m\xi^2}{\bar{\lambda}_\theta^2} - 2i\nu\Omega\lambda_z\xi^2\right)\right\}C$$
$$+\left\{\left[m\left(\Omega^2 - S\xi^2 + \frac{G-\alpha_{11}}{\bar{\lambda}_\theta^2}\right) + 2i\nu\Omega\lambda_z\right]\xi\zeta g(\zeta)\right.$$
$$\left. + \left(\frac{\alpha_{12}m}{\bar{\lambda}_\theta^2} - 2i\nu\Omega\lambda_z\right)\xi\zeta\right\}D = 0, \tag{3.8}$$

$$\{[m(S+\alpha_{21}-G)\xi^3\bar{\lambda}_\theta^{-2} - 2i\nu\Omega\lambda_z\xi^3]f(\xi) + m(\Omega^2 - \alpha_{22}\xi^2)\xi\bar{\lambda}_\theta^{-2}\}C$$
$$+\{[\lambda_z\Omega^2 - 2i\nu\Omega\lambda_z\xi^2 + m(S+\alpha_{21}-G)\xi^2\bar{\lambda}_\theta^{-2}]\zeta g$$
$$+m(\Omega^2 - \alpha_{22}\xi^2)\zeta\bar{\lambda}_\theta^{-2}\}D = 0, \tag{3.9}$$

where the following non-dimensionalized quantities are introduced

$$\beta = \rho c_0^2, \omega = c_0\Omega/R_0, \qquad N_z^0 = \beta hS, \qquad N_\theta^0 = \beta hG$$
$$\bar{\alpha}_{ij}^0 = \beta\alpha_{ij}, \qquad k = \xi/R_0, \qquad s = \zeta/R_0, \qquad \mu = \bar{\rho}c_0R_0\nu$$
$$m = \rho H/(\bar{\rho}R_0), \qquad \bar{\lambda}_\theta = r_0/R_0, \qquad f(\xi) = I_1(\xi\bar{\lambda}_\theta)/\{\xi\bar{\lambda}_\theta I_0(\xi\bar{\lambda}_\theta)\}$$
$$g(\zeta) = J_1(\zeta\bar{\lambda}_\theta)/\{\zeta\bar{\lambda}_\theta J_0(\zeta\bar{\lambda}_\theta)\},\ \ \bar{C} = c_0C/\{\bar{\lambda}_\theta I_0(\bar{\lambda}_\theta\xi)\},\ \ \bar{D} = c_0D/\{\bar{\lambda}_\theta J_0(\zeta\bar{\lambda}_\theta)\}. \tag{3.10}$$

In order for us to have a non-trivial solution for the unknown constants C and D, the determinant of the coefficient matrix must vanish. If this operation is carried out, the result will be as follows

$$(A_1\xi^2 + A_2)\Omega^4 + (A_3\xi^2 + A_4\xi^4)\Omega^2 + A_5\xi^4 + A_6\xi^6 = 0, \tag{3.11}$$

where the coefficients $A_i (i = 1, 2, \ldots, 6)$ are defined by

$$A_1 = m\bar{\lambda}_\theta^2[m(f-g) + \bar{\lambda}_\theta^2\lambda_z fg], A_2 = \bar{\lambda}_\theta^2\lambda_z(m + \lambda_z\bar{\lambda}_\theta^2 g),$$

$$A_3 = m\bar{\lambda}_\theta^2[m(S+\alpha_{22})(g-f) - \bar{\lambda}_\theta^2\lambda_z Sfg] + 4\nu^2\bar{\lambda}_\theta^4\lambda_z^2(f-g),$$

$$A_4 = 2i\nu\Omega\bar{\lambda}_\theta^2\lambda_z[m(f-g) + \lambda_z\bar{\lambda}_\theta^2 g(f-2)] + m^2(f-g)(G-\alpha_{11})$$
$$+ m\bar{\lambda}_\theta^2\lambda_z[fg(G-\alpha_{11}) + g(S-G+\alpha_{12}+\alpha_{21}) - \alpha_{22}],$$

$$A_5 = m^2\bar{\lambda}_\theta^2\alpha_{22}S(f-g),$$

$$A_6 = (f-g)(2i\nu\Omega m\bar{\lambda}_\theta^2\lambda_z(S-G+\alpha_{12}+\alpha_{21}-\alpha_{22}) + m^2[\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21}$$
$$+ (G-S)\alpha_{12} - G\alpha_{22}]). \tag{3.12}$$

Equation (3.11) gives the general dispersion relation of two distinct waves propagating in the medium. In general, both of these waves are dispersive. In blood-flow problems it is well-known that the Womersley parameter, $\alpha_0 = (\Omega/\nu)^{1/2}$, satisfies the condition $|\xi|^2\nu/\Omega \ll 1$ (Kuiken [12] and Bauer *et al*. [18]) and, hence, we may approximate ζ by $(-i\Omega/\nu)^{1/2} = i^{3/2}\alpha_0$. Therefore, we may neglect the terms with factors $\nu\Omega$ and $\nu^2\Omega^2$ appearing in (3.12). If this is done, the dispersion relation reduces to

$$(B_1\xi^2 + B_2)\Omega^4 + (B_3\xi^2 + B_4\xi^4)\xi^2\Omega^2 + (B_5\xi^2 + B_6)\xi^4 = 0, \tag{3.13}$$

where the coefficients $B_i (i = 1, 2, \ldots, 6)$ are defined by

$$B_1 = m\bar{\lambda}_\theta^2[m(f-g) + \bar{\lambda}_\theta^2\lambda_z fg], B_2 = \bar{\lambda}_\theta^2\lambda_z(m + \lambda_z\bar{\lambda}_\theta^2 g),$$

$$B_3 = m\bar{\lambda}_\theta^2[m(S+\alpha_{22})(g-f) - \bar{\lambda}_\theta^2\lambda_z Sfg],$$

$$B_4 = m^2(f-g)(G-\alpha_{11}) + m\bar{\lambda}_\theta^2\lambda_z[gf(G-\alpha_{11}) + g(S-G+\alpha_{12}+\alpha_{21}] - \alpha_{22},$$

$$B_5 = m^2\bar{\lambda}_\theta^2\alpha_{22}S(f-g),$$

$$B_6 = m^2[\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21} + (G-S)\alpha_{12} - G\alpha_{22}](f-g). \tag{3.14}$$

It is seen from (3.13) that the cut-off frequencies of these waves vanish. Introducing the non-dimensionalized phase velocity as $c = \Omega/\xi$ and decomposing it into real and imaginary parts as

$$c = X + iY, \tag{3.15}$$

we may express the propagation velocity v and the transmission coefficient χ as follows:

$$v = \Omega/\mathrm{Re}(\xi) = X^2 + Y^2/X$$

$$\chi = \exp[-2\pi\,\mathrm{Im}(\xi)/\mathrm{Re}(\xi)] = \exp(-2\pi Y/X). \tag{3.16}$$

Then the dispersion relation takes the following form

$$(B_1\xi^2 + B_2)c^4 + (B_3\xi^2 + B_4)c^2 + B_5\xi^2 + B_6 = 0. \tag{3.17}$$

Before we study the general dispersion relation, it will be instructive to examine some special cases.

3.1. Long-wave limit

For waves propagating in arteries the wavelength is, generally, very large in comparison with the mean radius of the arteries. Thus we may assume that $|\xi| \ll 1$ for which case $f(\xi) \to \frac{1}{2}$ and the dispersion relation takes the following form

$$D_0 c^4 + D_1 c^2 + D_2 = 0, \tag{3.18}$$

where the coefficients $D_i (i = 0, 1, 2)$ are defined by

$$D_0 = \bar{\lambda}_\theta^2 \lambda_z (m + \bar{\lambda}_\theta^2 \lambda_z g), \quad \zeta = i^{3/2} \alpha_0, \quad \alpha_0 = (\Omega/\nu)^{1/2},$$

$$D_1 = m^2 (\tfrac{1}{2} - g)(G - \alpha_{11}) + m \bar{\lambda}_\theta^2 \lambda_z [\tfrac{1}{2} g (G - \alpha_{11}) + g(S - G + \alpha_{12} + \alpha_{21}) - \alpha_{22}],$$

$$D_2 = m^2 (\tfrac{1}{2} - g)[\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21} + (G - S)\alpha_{12} - G\alpha_{22}]. \tag{3.19}$$

In addition, if the frequency is very low, then the function $g(\zeta)$ also approaches 1/2 and the dispersion relation (3.18) reduces to

$$(m + \tfrac{1}{2}\lambda_2 \bar{\lambda}_\theta^2) c^4 + m[(\tfrac{1}{4}(G - \alpha_{11})) + \tfrac{1}{2}(S - G + \alpha_{12} + \alpha_{21}) - \alpha_{22}] c^2 = 0. \tag{3.20}$$

The roots of this equation are given by

$$(c_0^2)_1 = 0, \quad (c_0^2)_2 = m(G + \alpha_{11} + 4\alpha_{22} - 2S - 2\alpha_{12} - 2\alpha_{21}) / \{2(\bar{\lambda}_\theta^2 \lambda_z + 2m)\}. \tag{3.21}$$

Particularly, if the initial deformation vanishes, i.e. $S = G = 0, \alpha_{11} = 4, \alpha_{12} = \alpha_{21} = 2, \alpha_{22} = 4$ and $\lambda_z = 1, \bar{\lambda}_\theta = 1$, (3.21) takes the following form

$$(c_0^2)_1 = 0, (c_0^2)_2 = 6m/(1 + 2m) = 6m + O(m^2). \tag{3.22}$$

This shows that at very low frequencies the viscosity does not effect the phase velocity significantly.

3.2. Inviscid fluid approximation

If the viscosity of the fluid is very small, we may approximate it as inviscid. In this particular case, we may obtain the dispersion relation from (3.13) and (3.14) by setting $\alpha_0 = (\Omega/\nu)^{1/2} \to \infty$ and $g(\zeta) \to 0$, *i.e.*,

$$\bar{\lambda}_\theta^2 (m f \xi^2 + \lambda_z) c^4 - [m \bar{\lambda}_\theta^2 f (S + \alpha_{22}) \xi^2 + \bar{\lambda}_\theta^2 \lambda_z \alpha_{22} + m f (\alpha_{11} - G)] c^2$$
$$+ m f [\bar{\lambda}_\theta^2 \alpha_{22} S \xi^2 + \alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21} + (G - S)\alpha_{12} - G\alpha_{22}] = 0. \tag{3.23}$$

When the viscosity of the fluid vanishes, the wave preserves its dispersive character. In particular and in addition to this, if the wavelength is very large as compared to the mean radius, we have from (3.23)

$$\lambda_z \bar{\lambda}_\theta^2 c^4 - [\bar{\lambda}_\theta^2 \lambda_z \alpha_{22} + \tfrac{1}{2} m (\alpha_{11} - G)] c^2 + \tfrac{1}{2} m [\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21} + (G - S)\alpha_{12} - G\alpha_{22}] = 0. \tag{3.24}$$

Furthermore, if the initial deformation vanishes, *i.e.* $\alpha_{11} = 4, \alpha_{12} = \alpha_{21} = 2, \alpha_{22} = 4, S = G = 0, \bar{\lambda}_\theta = 1, \lambda_2 = 1$, then the above equation reduces to

$$c^4 - 2(m+2)c^2 + 6m = 0. \tag{3.25}$$

The roots of this quadratic equation are given by

$$(c^2)_{1,2} = (2+m) \pm (m^2 - 2m + 4)^{1/2}. \tag{3.26}$$

Since for thin tubes the parameter m is very small (3.26) may be approximated by

$$c_1^2 = 4 + O(m), \quad c_2^2 = \frac{3}{2}m + O(m^2), \tag{3.27}$$

or, in terms of the real physical quantities, we have

$$v_1^2 = \beta c_1^2/\rho = 4\beta/\rho, \qquad v_2^2 = \beta c_2^2/\rho = 3\beta H/(2\bar{\rho} R_0). \tag{3.28}$$

For an isotropic, incompressible and elastic material the constant β is related to Young's modulus E through $\beta = E/3$, then the above wave speeds become

$$v_1^2 = 4E/(3\rho), \quad v_2^2 = EH/(2\bar{\rho} R_0). \tag{3.29}$$

Of these wave speeds, the first one corresponds to the Lamb and the second to the Young (Moens-Korteweg) modes, respectively.

4. Numerical analysis and discussion

Having investigated some special cases by analytical means, we can now study the more general case by a numerical approach. To this end, we need explicit expressions for F and G, and the numerical values of m and α. For the membrane approximations $\lambda_\theta = \bar{\lambda}_\theta = r_0/R_0$ and $t_{rr}^0 \cong 0$, thus from (2.9) we have

$$S = (\bar{\lambda}_\theta^2 \lambda_z^2 - \lambda_z^{-2}) F(\bar{\lambda}_\theta), \qquad G = (\bar{\lambda}_\theta^2 \lambda_z^2 - \bar{\lambda}_\theta^{-2}) F(\bar{\lambda}_\theta),$$

$$F(\bar{\lambda}_\theta) = \exp[\alpha(\bar{\lambda}_\theta^{-2} + \lambda_z^{-2} + \bar{\lambda}_\theta^2 \lambda_z^2 - 3)], \qquad P_i = G/(R_0 \bar{\lambda}_\theta), m = \rho H/(\bar{\rho} R_0). \tag{4.1}$$

From (2.35) the non-dimensionalized coefficients α_{ij} may be given by

$$\alpha_{11} = F(\lambda_\theta)[(\lambda_\theta^2 \lambda_z^2 + 3\lambda_\theta^{-2}) + 2\alpha(\lambda_\theta^2 \lambda_z^2 - \lambda_\theta^{-2})^2]$$

$$\alpha_{12} = F(\lambda_\theta)[(\lambda_\theta^2 \lambda_z^2 + \lambda_\theta^{-2}) + 2\alpha(\lambda_\theta^2 \lambda_z^2 - \lambda_z^{-2})(\lambda_\theta^2 \lambda_z^2 - \lambda_\theta^{-2})]$$

$$\alpha_{21} = F(\lambda_\theta)[(\lambda_\theta^2 \lambda_z^2 + \lambda_z^{-2}) + 2\alpha(\lambda_\theta^2 \lambda_z^2 - \lambda_z^{-2})(\lambda_\theta^2 \lambda_z^2 - \lambda_\theta^{-2})]$$

$$\alpha_{22} = F(\lambda_\theta)[(\lambda_\theta^2 \lambda_z^2 + 3\lambda_z^2) + 2\alpha(\lambda_\theta^2 \lambda_z^2 - \lambda_z^{-2})^2] \tag{4.2}$$

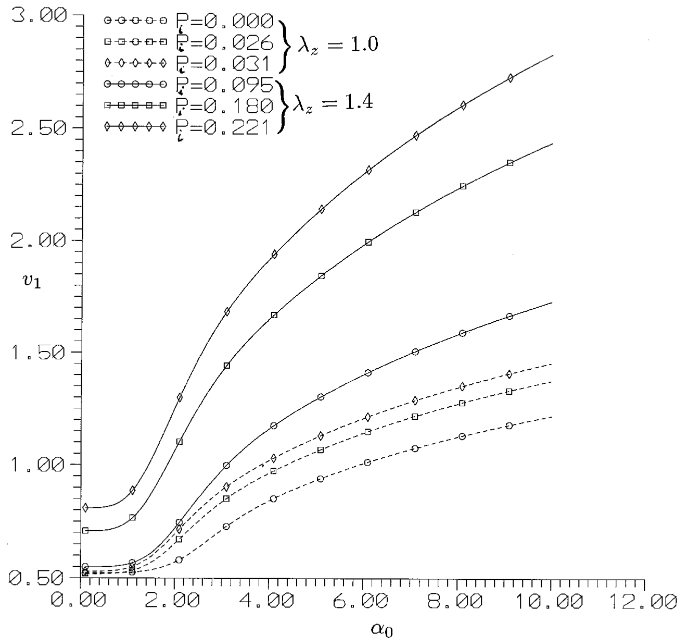

Figure 2. Variation of primary wave speed with Womersley parameter, inner pressure and axial stretch ratio.

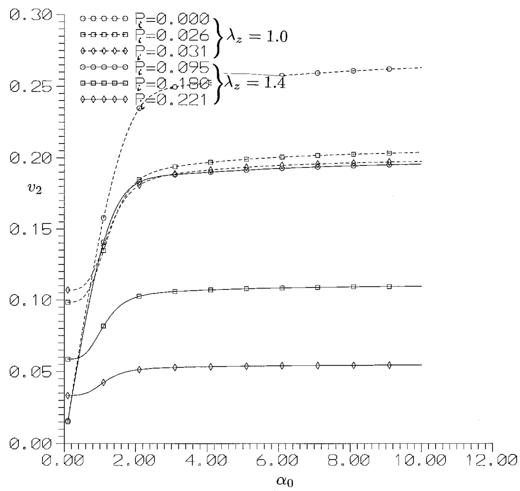

Figure 3. Variation of secondary wave speed with Womersley parameter, inner pressure and axial stretch ratio.

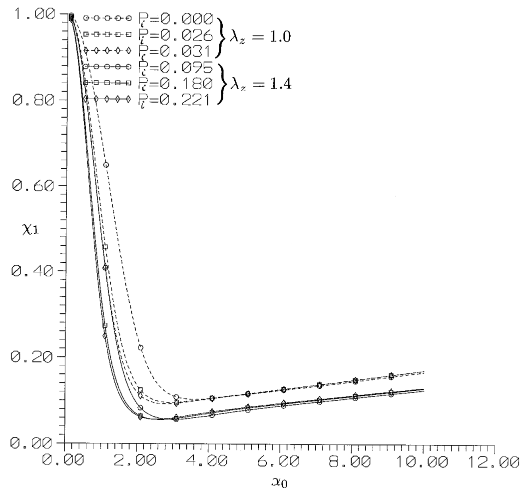

Figure 4. Variation of transmission coefficient of primary wave with Womersley parameter, inner pressure and axial stretch ratio.

Using the experimental results by Simon *et al*. [15] on canine abdominal arteries, we have previously determined the value of coefficient α as 0·82. Employing this value of α and noticing that $\rho/\bar{\rho} \cong 1$, we analyzed the dispersion relation numerically for $m = 0{\cdot}05$ and the results are depicted in Figures 2–5. Figure 2 gives the variation of the speed of the primary

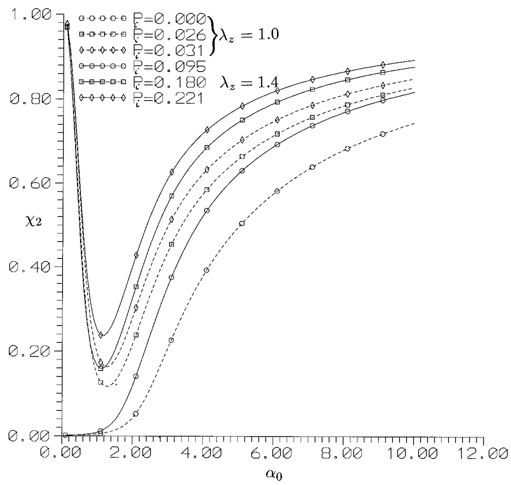

Figure 5. Variation of transmission coefficient of secondary wave with Womersley parameter, inner pressure and axial stretch ratio.

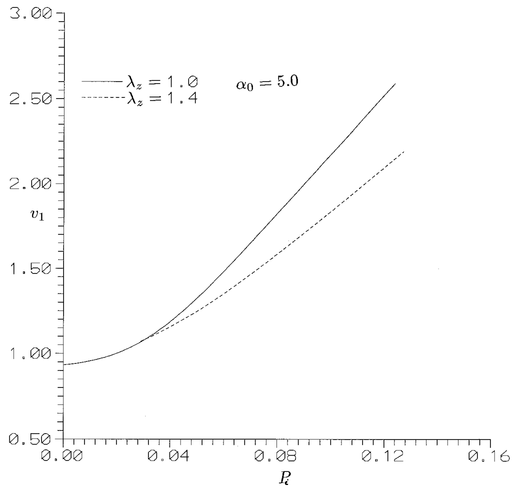

Figure 6. Variation of primary wave speed with inner pressure and axial stretch ratio.

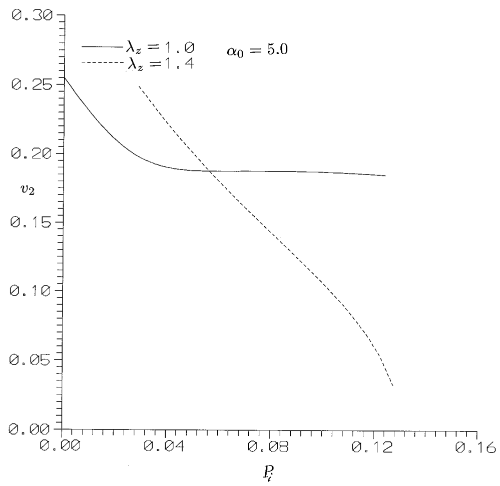

Figure 7. Variation of secondary wave speed with inner pressure and axial stretch ratio.

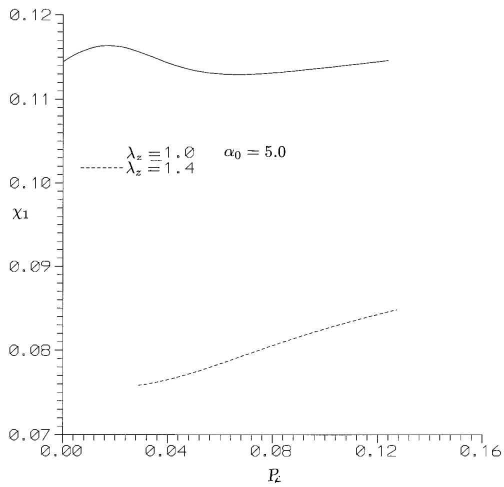

Figure 8. Variation of transmission coefficient of primary wave with inner pressure and axial stretch ratio.

wave (Lamb mode) with Womersley parameter α_0, inner pressure and axial stretch ratio. The numerical evaluation indicates that the wave speed increases with Womersley parameter and inner pressure, but decreases with axial stretch ratio (see also Figure 6). The result is consistent with the findings of Erbay *et al*. [19], who employed a strain-energy density function applicable to soft biological tissues, but differs from those of Atabek and Lew [8], Kuiken [12] and Demiray and Ercengiz [20], who utilized the constitutive relations applicable to engineering materials. The main factor for such a different behavior is that soft biological tissues get stiffer under large deformation, whereas the engineering materials get softer with

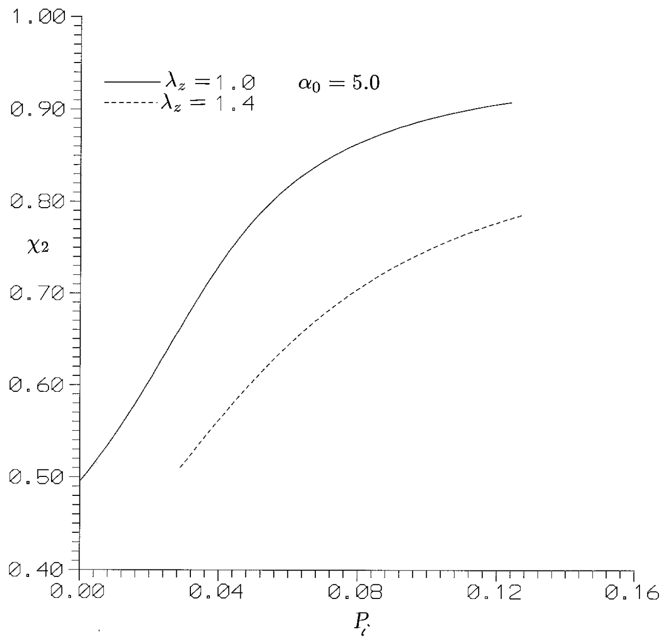

Figure 9. Variation of transmission coefficient of secondary wave with inner pressure and axial stretch ratio.

increasing deformation. In other words, because of the the different inner structure of biological materials, the tangent modulus increases with inner pressure, while the tangent modulus for engineering materials decreases with deformation. The variation of the speed of the secondary wave (Young mode) with Womersley parameter α_0, inner pressure and the axial stretch ratio is shown in Figure 3. This figure reveals that the speed of the secondary wave increases with Womersley parameter and stretch ratio, but decreases with inner pressure (see also Figure 7). However, after a certain value of the Womersley parameter, the speed becomes insensitive to variations of this parameter. This result again is consistent with the result of [19], but differs from other works that employed the constitutive relation of engineering materials. As shown in Figure 4, the transmission coefficient of the primary wave first decreases very rapidly with Womersley parameter and then increases steadily with this parameter. This coefficient increases with inner pressure, but decreases with the axial stretch ratio (see also Figure 8). Finally, the variation of the transmission coefficient of the secondary wave is depicted in Figure 5. This coefficient first decreases very rapidly with Womersley parameter and then starts to increases with both the same parameter and the inner radius, but decreases with axial stretch ratio (see also Figure 9).

In conclusion, by employing the large deformation analysis and the theory of ‘small deformations superimposed on large initial static deformations’ for an elastic thin cylindrical shell and the linearized equations of a viscous fluid, the propagation of harmonic waves in two such interacting media is studied and the dependence of propagation velocities and transmission coefficients on the initial deformation, material and geometrical characteristics is obtained. This nonlinear theory makes it possible to trace the variation of propagation characteristics with changes in blood pressure, the mechanical properties and the geometrical characteristics of the arterial wall material. Such observations can be used in the diagnosis of some circulatory mulfunctionings.

In studying this problem, we have employed the linearized equations of thin shells, which are valid for $H/R_0 \leqslant 0{\cdot}1$. However, even for large arteries, this ratio changes between $1/6 \sim 1/4$ and, hence, the thin-shell theories cannot be applied in the case of arterial mechanics. Thus, for a better understanding of wave propagation in arteries, the present formulation should be extended to thick-shell theories. As a final remark, we should point out that the analysis of nonlinear field equations for the tube and the fluid will be another interesting subject for future study.

Acknowledgement

This work is partially supported by the Turkish Academy of Sciences and TÜBITAK, Mechanics and Applied Mathematics Research Unit.

References

1. D.A. McDonald, *Blood Flow in Arteries*, Baltimore: Williams and Wilkins (1966) pp. 496.
2. Y.C. Fung, *Biodynamics: Circulation.* New York: Springer Verlag (1984) pp. 404.
3. P. Lambossy, Aperçu et historique sur le problème de la propagation des ondes dans un liquide compressible enfermé dans un tube élastique. *Helv. Physiol. Acta* 9 (1951) 145–161.
4. R. Skalak, Wave propagation in blood flow. In: *Biomechanics Symposium*, Y. C. Fung (ed.), pp. 20–46, ASME, New York (1966).
5. K. Witzig, Über erzwungene Wellenbewegungen Zaher, inkompressibler Flüssigkeiten in elastischen Rohren. *Inaugural Dissertation, University of Bern* (1914).
6. G.W. Morgan and J.P. Kiely, Wave propagation in a viscous liquid contained in a flexible tube. *J. Acoust. Soc. Am.* 26 (1954) 323–328.
7. J.R. Womersley, Oscillatory motion of a viscous liquid in a thin walled elastic tube. I: The linear approximation for long waves. *Phil. Mag.* 46 (1955) 199–219.
8. H.B. Atabek and H.S. Lew, Wave propagation through a viscous incompressible fluid contained in an initially stressed elastic tube. *Biophysical J.* 6 (1966) 481–503.
9. I. Mirsky, Wave propagation in a viscous fluid contained in an orthotropic elastic tube. *Biophysical J.* 7 (1967) 165–186.
10. H.B. Atabek, Wave propagation through a viscous fluid contained in a tethered, initially stressed orthotropic elastic tube. *Biophysical J.* 8 (1968) 626–649.
11. A.I. Rachev, Effect of transmural and muscular activity on pulse waves in arteries. *J. Biomech. Engng* 102 (980) 119–123.
12. G.D.C. Kuiken, Wave propagation in a thin walled liquid-filled initially stressed tube. *J. Fluid Mech.* 141 (1984) 289–308.
13. H. Demiray, Stresses in ventricular wall. *J. Appl. Mech., Trans. ASME* 98 (1976) 194–197.
14. R.H. Cox, Anisotropic properties of the canine carotid artery *in vitro. J. Biomechanics* 8 (1975) 293–300.
15. B.R. Simon, A.S. Kobayashi, D.E. Strandness and C.A. Wiederhielm, Re-evaluation of arterial constitutive laws. *Circul. Res.* 30 (1952) 491–500.
16. A.E. Green and W. Zerna, *Theoretical Elasticity.* Oxford: Clarendon Press (1968) pp. 457.
17. A.C. Eringen and E.S. Şuhubi, *Elastodynamics, Vol.I.* New York: Academic Press (1974) pp. 341.
18. H.F. Bauer, S.T. Metten and J. Seikmann, Dynamical behavior of distensible fluid lines carrying a pulsating incompressible liquid. *ZAMM* 60 (1980) 221–234.
19. H.A. Erbay, S. Erbay and H. Demiray, Effect of prestress on pulse waves in arteries. *ZAMM* 67 (1987) 473–485.
20. H. Demiray and A. Ercengiz, Wave propagation in a prestressed elastic tube filled with a viscous fluid. *Int. J. Engng. Sci* 29 (1991) 575–585.